Conditions for the Local Boundedness of Solutions of the Navier-Stokes System in Three Dimensions

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Theorem. *Let* $\Omega \subseteq \mathbf{R}^3$, *and let* $\mathbf{v} \in L_\infty(0,T;L_2(\Omega)) \cap L_2(0,T;W_2^1(\Omega))$ *be a weak solution of the Navier–Stokes system*

$$\begin{aligned} \mathbf{v}_t - \triangle \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p &= 0, \\ \operatorname{div} \mathbf{v} &= 0. \end{aligned}$$

Suppose that there is some $\epsilon > 0$ *so that either*

$$\operatorname*{ess\,sup}_{(x,t)\in\Omega_T} \sup_{\rho>0} \frac{1}{\rho^{\frac{5}{3}+\epsilon}} \iint_{Q_\rho(x,t)} |\mathbf{v}(\xi,\tau)|^{\frac{10}{3}} \, d\xi \, d\tau < \infty,$$

or

$$\operatorname*{ess\,sup}_{(x,t)\in\Omega_T} \sup_{\rho>0} \frac{1}{\rho^{1+\epsilon}} \iint_{Q_\rho(x,t)} |\nabla \mathbf{v}(\xi,\tau)|^2 \, d\xi \, d\tau < \infty,$$

where $Q_\rho(x,t) = [B_\rho(x) \times (t-\rho^2, t)] \cap \Omega_T$ *and* $\Omega_T = \Omega \times (0,T)$.

Then $\mathbf{v} \in L_{\infty,loc}(\Omega_T)$, *and* $\mathbf{v}$ *is* C^∞ *in the spatial variables.*

Theorem. *Let* $\Omega \subseteq \mathbf{R}^3$*, and let* $\mathbf{v} \in L_\infty(0,T;L_2(\Omega)) \cap L_2(0,T;W_2^1(\Omega))$ *be a weak solution of the Navier–Stokes system. Suppose that either*

i) There is some $2 < q < 5$ *and some* $\lambda > 5 - q$ *that*

$$\operatorname*{ess\,sup}_{(x,t)\in\Omega_T} \sup_{\rho>0} \frac{1}{\rho^\lambda} \iint_{Q_\rho(x,t)} |\mathbf{v}(\xi,\tau)|^q \, d\xi \, d\tau < \infty,$$

or

ii) There is some $10/7 < q \leq 5/2$ *and* $\lambda > 5 - 2q$ *so that*

$$\operatorname*{ess\,sup}_{(x,t)\in\Omega_T} \sup_{\rho>0} \frac{1}{\rho^\lambda} \iint_{Q_\rho(x,t)} |\nabla\mathbf{v}(\xi,\tau)|^q d\xi d\tau < \infty,$$

where $Q_\rho(x,t) \equiv [B_\rho(x) \times (t-\rho^2,t)] \cap \Omega_T$.

Then $\mathbf{v} \in L_{\infty,loc}(\Omega_T)$ *and* $\mathbf{v}$ *is* C^∞ *in the spatial variables.*

Significance of the result

- Generalizes the local regularity result of Serrin.
- Similar in spirit to the partial regularity result of Caffarelli, Kohn and Nirenberg.
- True for all weak solutions.
  - Solutions do not need to solve an initial-boundary value problem with zero boundary data.
  - Solutions do not need to be suitable weak solutions.
- The degree of smoothness cannot be simply improved. Consider $\mathbf{v}(x,t) = a(t)\nabla\psi(x)$ for some harmonic function $\psi(x)$ and some merely integrable function $a(t)$.

Theorem (Serrin, Takahashi). *Let* $\Omega \subseteq \mathbf{R}^N$ *and let* $\mathbf{v} \in L_\infty(0,T;L_2(\Omega)) \cap L_2(0,T;W_2^1(\Omega))$ *be a weak solution of the Navier-Stokes system*

$$\begin{aligned} \mathbf{v}_t - \triangle \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p &= 0 \\ \operatorname{div} \mathbf{v} &= 0 \end{aligned}$$

in Ω_T*. If either*

i) $\mathbf{v} \in L_q(0,T;L_r(\Omega))$ *for* q *and* r *with* $2/q + N/r = 1$, $N < r \le \infty$, *or*

ii) $\operatorname{ess\,sup}_{0<t<T} \|\mathbf{v}(\cdot,t)\|_{L_N(\Omega)}$ *is sufficiently small*

Then $\mathbf{v} \in L_{\infty,loc}(\Omega_T)$ *and* $\mathbf{v}$ *is* C^∞ *in the spatial variables.*

The following is well-known.

Theorem. *Let* $\Omega \subseteq \mathbf{R}^N$ *be smooth, and let* $\mathbf{v} \in L_\infty(0,T; L_2(\Omega)) \cap L_2(0,T; W_2^1(\Omega))$ *be a weak solution of the Navier-Stokes initial-boundary value problem*

$$
\begin{aligned}
\mathbf{v}_t - \triangle \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p &= 0 \\
\operatorname{div} \mathbf{v} &= 0 \\
\mathbf{v}\big|_{\partial\Omega \times (0,T)} &= 0 \\
\mathbf{v}\big|_{t=0} &= \mathbf{v}_o \in H(\Omega)
\end{aligned}
$$

If either

i) $\mathbf{v} \in L_q(0,T; L_r(\Omega))$ *for* q *and* r *with* $2/q + N/r = 1$, $N < r \leq \infty$, *or*

ii) $\mathbf{v} \in C^0([0,T]; L_N(\Omega))$

Then $\mathbf{v} \in C^\infty(\bar{\Omega} \times (0,T])$.

Theorem (Caffarelli, Kohn, Nirenberg). *There is a constant* $\delta > 0$ *so that if* $\mathbf{v}$ *is a suitable weak solution of the Navier-Stokes system and*

$$\limsup_{\rho \downarrow 0} \frac{1}{\rho} \iint_{Q^*_\rho(x,t)} |\nabla \mathbf{v}(\xi, \tau)|^2 \, d\xi \, d\tau < \delta$$

then $\mathbf{v}$ *is essentially bounded in a neighborhood of* (x,t)*, where* $Q^*_\rho(x,t) = B_\rho(x) \times (t - \frac{7}{8}\rho^2, t + \frac{1}{8}\rho^2)$.

- If $\mathbf{v}$ is essentially bounded in a neighborhood of (x,t), we say that (x,t) is a *regular point*, otherwise it is a *singular point*.
- This result implies that the set of singular points has Hausdorff dimension of at most 1.
- A suitable weak solution satisfies some additional conditions, most significant of which is a generalized energy inequality.

The Proof

There are two main elements of the proof-

- A local representation theorem, and
- A generalization of the Hardy-Littlewood-Sobolev theorem on fractional integration.

Proposition. *Let $\Omega \subseteq \mathbf{R}^3$, and let $\mathbf{v} \in L_\infty(0,T;L_2(\Omega)) \cap L_2(0,T;W_2^1(\Omega))$ be a weak solution of the Navier–Stokes system*

$$\begin{aligned} \mathbf{v}_t - \triangle \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p &= 0, \\ \operatorname{div} \mathbf{v} &= 0. \end{aligned}$$

There exists an absolute constant γ so that for almost every $(x,t) \in \Omega_T$ with $Q_R(x,t) \subset\subset \Omega_T$,

$$\begin{aligned} |\mathbf{v}(x,t)| \leq \gamma & \fint\!\!\fint_{Q_R \setminus Q_{R/2}(x,t)} |\mathbf{v}(\xi,\tau)|\, d\xi\, d\tau \\ & + \gamma \iint_{Q_R(x,t)} \frac{|\mathbf{v}(\xi,\tau)|^2}{(|x-\xi| + \sqrt{t-\tau})^4} d\xi\, d\tau \end{aligned}$$

and

$$\begin{aligned} |\mathbf{v}(x,t)| \leq \gamma & \fint\!\!\fint_{Q_R \setminus Q_{R/2}(x,t)} |\mathbf{v}(\xi,\tau)|\, d\xi\, d\tau \\ & + \gamma \iint_{Q_R(x,t)} \frac{|[(\mathbf{v}\cdot\nabla)\mathbf{v}](\xi,\tau)|}{(|x-\xi| + \sqrt{t-\tau})^3} d\xi\, d\tau. \end{aligned}$$

Proposition. *Let $\mathcal{V} \subset \mathbf{R}^N \times \mathbf{R}$ be a bounded domain, and suppose that $f \in L_m(\mathcal{V})$ with*

$$\operatorname*{ess\,sup}_{(x,t)\in\mathcal{V}} \sup_{\rho>0} \frac{1}{\rho^\lambda} \iint_{Q_\rho(x,t)} |f|^q \, d\xi \, d\tau \equiv |f|^q_{\mathfrak{L}^\lambda_q(\mathcal{V})} < \infty$$

for some $m \geq q > 1$ and $0 \leq \lambda < N + 2$.

For $(x,t) \in \mathcal{V}$, define

$$Tf(x,t) = \iint_{\mathcal{V}} \frac{f(\xi,\tau)}{(|x-\xi| + \sqrt{|t-\tau|})^{N+2-\alpha}} d\xi \, d\tau.$$

Then for any $m < p < \infty$ satisfying

$$\frac{1}{p} > \frac{q}{m}\left(\frac{1}{q} - \frac{\alpha}{N+2-\lambda}\right)$$

there is a constant $\gamma = \gamma(N,p,q,m,\alpha,\lambda,\mathcal{V})$ so that

$$\|Tf\|_{L_p(\mathcal{V})} \leq \gamma \|f\|^{\frac{m}{p}}_{L_m(\mathcal{V})} |f|^{1-\frac{m}{p}}_{\mathfrak{L}^\lambda_q(\mathcal{V})}.$$

Proposition. *Let $\mathcal{V} \subset \mathbf{R}^N \times \mathbf{R}$ be a bounded domain, suppose that $f \in L_m(\mathcal{V})$, and*

$$\operatorname*{ess\,sup}_{(x,t)\in\mathcal{V}} \sup_{\rho>0} \frac{1}{\rho^\lambda} \iint_{Q_\rho(x,t)} |g|^q \, d\xi \, d\tau \equiv |g|^q_{\mathfrak{L}^\lambda_q(\mathcal{V})} < \infty,$$

for some m and q with $1/m + 1/q < 1$ and some $0 \le \lambda < N+2$.

For $(x,t) \in \mathcal{V}$, define

$$T(f,g)(x,t) = \iint_{\mathcal{V}} \frac{f(\xi,\tau)g(\xi,\tau)}{\left(|x-\xi| + \sqrt{|t-\tau|}\right)^{N+2-\alpha}} d\xi d\tau.$$

Then for any $m < p < \infty$ satisfying

$$\frac{1}{p} > \frac{1}{m} + \frac{1}{q} - \frac{\alpha + \frac{\lambda}{q}}{N+2}$$

there is a constant $\gamma = \gamma(N,p,q,m,\alpha,\lambda,\mathcal{V})$ so that

$$\|T(f,g)\|_{L_p(\mathcal{V})} \le \gamma \|f\|_{L_m(\mathcal{V})} |g|_{\mathfrak{L}^\lambda_q(\mathcal{V})}.$$

Proof of the Main Result

The representation theorem implies

$$|\mathbf{v}(x,t)| \leq \gamma \mathop{⨏\!\!\!\!⨏}_{Q_R(x,t)} |\mathbf{v}(\xi,\tau)|\, d\xi\, d\tau + \gamma \iint_{\mathcal{V}} \frac{|\mathbf{v}(\xi,\tau)|^2}{(|x-\xi| + \sqrt{|t-\tau|})^4}\, d\xi\, d\tau.$$

for any $\mathcal{V} \subset\subset \Omega_T$.

The first of these is bounded uniformly.

As for the second, suppose that $\mathbf{v} \in L_m(\mathcal{V})$. Our hypotheses require $\| |\mathbf{v}|^2 |_{\mathfrak{L}^{\lambda}_{q/2}} < \infty$, so our fractional integration result implies that the second term is in $L_p(\mathcal{V})$ for any

$$p < \frac{m}{2 - \frac{q}{5-\lambda}}.$$

Proceed by induction until we can apply the result of Serrin.

In the second case, the representation theorem implies

$$|\mathbf{v}(x,t)| \leq \gamma \fint\!\!\fint_{Q_R(x,t)} |\mathbf{v}(\xi,\tau)|\, d\xi\, d\tau + \gamma \iint_{\mathcal{V}} \frac{|\mathbf{v}(\xi,\tau)||\nabla \mathbf{v}(\xi,\tau)|}{(|x-\xi| + \sqrt{|t-\tau|})^3}\, d\xi\, d\tau.$$

Suppose that $\mathbf{v} \in L_m(\mathcal{V})$. Then the singular integral is in $L_p(\mathcal{V})$ for

$$\frac{1}{p} > \frac{1}{m} - \frac{1}{5q}(\lambda - (5-2q))$$

We proceed by induction until we can apply the result of Serrin.

Proof of the Representation Result

Recall the fundamental solution of the Stokes system centered at (x_o, t_o)

$$T_{jk}(x_o - x, t_o - t) = \delta_{jk}\Gamma(x_o - x, t_o - t) + \frac{1}{4\pi}\frac{\partial^2}{\partial x_j \partial x_k}\int_{\mathbf{R}^3}\frac{\Gamma(y, t_o - t)}{|(x_o - x) - y|}dy$$

where δ_{jk} is the Kronecker delta and

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{3/2}}\exp\left(-\frac{|x|^2}{4t}\right)$$

is the fundamental solution of the heat equation.

Set $\mathbf{T}_k = (T_{1k}, T_{2k}, T_{3k}) = (T_{k1}, T_{k2}, T_{k3})$.

Because $\mathbf{T}_k$ is solenoidal, $\triangle \mathbf{T}_k = -\operatorname{curl}\operatorname{curl}\mathbf{T}_k$, and we can use the Newtonian potential to write $\mathbf{T}_k$ as

$$\mathbf{T}_k(x_o - x, t_o - t) = \frac{1}{4\pi}\operatorname{curl}\int_{\mathbf{R}^3} \frac{\operatorname{curl}\mathbf{T}_k(\xi, t_o - t)}{|(x_o - x) - \xi|} d\xi.$$

We localize by using a cutoff function $\zeta(x,t)$ with $\zeta(x,t) = 1$ if $(x,t) \in Q_{R/2}(x_o, t_o)$, so that $\zeta(x,t) = 0$ if $(x,t) \notin Q_{3R/4}(x_o, t_o)$.

Our test function is then

$$\mathbf{\Phi}_k(x,t) = \frac{1}{4\pi}\operatorname{curl}\left\{\zeta(x,t)\int_{\mathbf{R}^3} \frac{\operatorname{curl}\mathbf{T}_k(\xi, t_o - t)}{|(x_o - x) - \xi|} d\xi\right\}.$$

Our test function can be written as

$$\begin{aligned}\mathbf{\Phi}_k(x,t) &= \zeta(x,t)\mathbf{T}_k(x_o - x, t_o - t) \\ &\quad + \frac{1}{4\pi}\nabla\zeta(x,t) \times \int_{\mathbf{R}^3} \frac{\operatorname{curl}\mathbf{T}_k(\xi, t_o - t)}{|(x_o - x) - \xi|} d\xi.\end{aligned}$$

Let $Q_R = Q_R(x_o, t_o)$. Then

$$\begin{aligned}&\iint_{Q_R} \mathbf{v}_\eta\cdot(\partial_t + \triangle)\left[\zeta(x,t)\mathbf{T}_k(x_o - x, t_o - t)\right]\,dx\,dt \\ &\qquad + \frac{1}{4\pi}\iint_{Q_R} \mathbf{v}_\eta\cdot(\partial_t + \triangle)\left\{\nabla\zeta(x,t)\right. \\ &\qquad\qquad \left.\times \int_{\mathbf{R}^3} \frac{\operatorname{curl}\mathbf{T}_k(\xi, t_o - t)}{|(x_o - x) - \xi|} d\xi\right\} dx\,dt \\ &\qquad = \iint_{Q_R} [(\mathbf{v}\cdot\nabla)\mathbf{v}]_\eta\cdot\mathbf{\Phi}_k\,dx\,dt \\ &\quad I + J = K\end{aligned}$$

where $\mathbf{v}_\eta$ is a mollification of $\mathbf{v}$.

Thus, if $\mathcal{H}u = u_t - \triangle u$, and $\mathcal{H}^* u = u_t + \triangle u$

$$I = -\iint_{Q_R} (\mathcal{H}\mathbf{v}_\eta)\cdot\zeta(x,t)\mathbf{T}_k(x_o - x, t_o - t)\,dx\,dt$$

$$= -\iint_{Q_R} \mathcal{H}[(v_k)_\eta\zeta]\Gamma(x_o - x, t_o - t)\,dx\,dt$$

$$+ \iint_{Q_R} (v_k)_\eta(\mathcal{H}\zeta)\Gamma(x_o - x, t_o - t)\,dx\,dt$$

$$- 2\sum_{j=1}^{3}\iint_{Q_R} \frac{\partial(v_k)_\eta}{\partial x_j}\frac{\partial\zeta}{\partial x_j}\Gamma(x_o - x, t_o - t)\,dx\,dt$$

$$- \frac{1}{4\pi}\sum_{j=1}^{3}\iint_{Q_R} (v_j)_\eta$$

$$\mathcal{H}^*\left\{\frac{\partial\zeta}{\partial x_j}\frac{\partial}{\partial x_k}\int_{\mathbf{R}^3}\frac{\Gamma(y, t_o - t)}{|(x_o - x) - y|}dy\right\}dx\,dt$$

Now

$$-\iint_{Q_R} \mathcal{H}[(v_k)_\eta\zeta]\Gamma(x_o - x, t_o - t)dx dt = (v_k)_\eta(x_o, t_o).$$

Further for any $t > 0$, for any integers $\ell, m \geq 0$ and for any spatial derivative of order m there exists a constant C depending only on ℓ and m so that

$$|D_t^\ell D_x^m \Gamma(x,t)| \leq \frac{C}{(|x| + \sqrt{t})^{3+m+2\ell}},$$

$$|D_t^\ell D_x^m T_{jk}(x,t)| \leq \frac{C}{(|x| + \sqrt{t})^{3+m+2\ell}},$$

$$|D_t^\ell D_x^m \int_{\mathbb{R}^3} \frac{\Gamma(y,t)}{|x-y|}\, dy| \leq \frac{C}{(|x| + \sqrt{t})^{1+m+2\ell}}.$$

Thus

$$|I - (v_k)_\eta(x_o, t_o)| \leq \frac{\gamma}{R^5} \iint_{Q_R \setminus Q_{R/2}} |\mathbf{v}_\eta|\, d\xi\, d\tau.$$

Now

$$\mathbf{T}_k(x,t) = \Gamma(x,t)\mathbf{e}_k + \frac{1}{4\pi}\,\mathrm{grad}\,\frac{\partial}{\partial x_k}\int_{\mathbb{R}^3}\frac{\Gamma(y,t)}{|x-y|}\,dy$$

so that

$$\mathrm{curl}\,\mathbf{T}_k(x,t) = \nabla\Gamma(x,t)\times\mathbf{e}_k,$$

and thus

$$J = \frac{-1}{4\pi}\iint_{Q_R}\mathcal{H}\mathbf{v}_\eta\cdot\Bigg\{\nabla\zeta\times \left[\nabla\int_{\mathbf{R}^3}\frac{\Gamma(\xi,t_o-t)}{|(x_o-x)-\xi|}\,d\xi\right]\times\mathbf{e}_k\Bigg\}dx\,dt.$$

Integrating by parts, we find

$$J = \frac{1}{4\pi} \iint_{Q_R} \mathbf{v}_\eta \cdot \Bigg\{ \mathcal{H} \nabla \zeta$$

$$\times \left[\nabla \int_{\mathbf{R}^3} \frac{\Gamma(\xi, t_o - t)}{|(x_o - x) - \xi|} \, d\xi \right] \times \mathbf{e}_k \Bigg\} dx \, dt$$

$$+ \frac{1}{4\pi} \iint_{Q_R} \mathbf{v}_\eta \cdot \Bigg\{ \nabla \zeta$$

$$\times \mathcal{H}^* \left[\nabla \int_{\mathbf{R}^3} \frac{\Gamma(\xi, t_o - t)}{|(x_o - x) - \xi|} \, d\xi \right] \times \mathbf{e}_k \Bigg\} dx \, dt$$

$$+ \frac{1}{2\pi} \sum_{\ell=1}^{3} \iint_{Q_R} \mathbf{v}_\eta \cdot \Bigg\{ \left(\frac{\partial}{\partial x_\ell} \nabla \zeta \right)$$

$$\times \frac{\partial}{\partial x_\ell} \left[\nabla \int_{\mathbf{R}^3} \frac{\Gamma(\xi, t_o - t)}{|(x_o - x) - \xi|} \, d\xi \right] \times \mathbf{e}_k \Bigg\} dx \, dt.$$

Hence the estimates of Γ imply

$$|J| \leq \frac{\gamma}{R^5} \iint_{Q_R \setminus Q_{R/2}} |\mathbf{v}_\eta| \, d\xi \, d\tau.$$

Finally

$$\begin{aligned} K = &\iint_{Q_R} [(\mathbf{v} \cdot \nabla)\mathbf{v}]_\eta \cdot \zeta \mathbf{T}_k(x_o - x, t_o - t) \, dx \, dt \\ &+ \frac{1}{4\pi} \iint_{Q_R} [(\mathbf{v} \cdot \nabla)\mathbf{v}]_\eta \cdot \Bigg\{ \nabla \zeta \\ &\times \left[\nabla \int_{\mathbf{R}^3} \frac{\Gamma(\xi, t_o - t)}{|(x_o - x) - \xi|} \, d\xi \right] \times \mathbf{e}_k \Bigg\} dx \, dt \end{aligned}$$

so that the estimates of $\mathbf{T}$ and Γ imply

$$|K| \leq \gamma \iint_{Q_R} \frac{|[(\mathbf{v} \cdot \nabla)\mathbf{v}]_\eta|}{(|x_o - x| + \sqrt{t_o - t})^3} dx \, dt.$$

Alternatively, we could integrate by parts first, and see

$$K = -\sum_{j,m=1}^{3} \iint_{Q_R} (v_m v_j)_\eta \frac{\partial}{\partial x_m}\big\{\zeta(x,t) T_{jk}(x_o - x, t_o - t)\big\}\, dx\, dt$$

$$-\sum_{j,m=1}^{3} \frac{1}{4\pi} \iint_{Q_R} (v_m v_j)_\eta \frac{\partial}{\partial x_m} \left\{ \nabla\zeta \times \left[\nabla \int_{\mathbf{R}^3} \frac{\Gamma(\xi, t_o - t)}{|(x_o - x) - \xi|}\, d\xi \right] \times \mathbf{e}_k \right\}_j dx\, dt$$

so that the estimates of $\mathbf{T}$ and Γ imply

$$|K| \le \gamma \iint_{Q_R} \frac{|[(\mathbf{v}\cdot\nabla)\mathbf{v}]_\eta|}{(|x_o - x| + \sqrt{t_o - t})^3} dx\, dt.$$

Passing to the limit as $\eta \downarrow 0$ proves the representation theorem.

Proof of the Fractional Integration Results

For $h \in L_{p'}(\mathcal{V})$ with $1/p + 1/p' = 1$,

$$\iint_{\mathcal{V}} h(x,t)\, Tf(x,t)\, dx\, dt$$
$$= \sum_{n=A_{\mathcal{V}}}^{\infty} \iint_{\mathcal{V}} \iint_{Q^n(x,t)} \frac{h(x,t)\, f(\xi,\tau)\, d\xi\, d\tau\, dx\, dt}{\left(|x-\xi| + \sqrt{|t-\tau|}\right)^{N+2-\alpha}}$$

where

$$Q^n(x,t) = \left\{(\xi,\tau) : |x-\xi| + \sqrt{t-\tau} \in [2^{-n-1}, 2^{-n}]\right\}$$

and $A = A_{\mathcal{V}}$ is an integer chosen so that $B_{2^{-A}}(x) \times (t - 2^{-2A}, t + 2^{-2A}) \supseteq \mathcal{V}$ for all $(x,t) \in \mathcal{V}$.

Now for each n, Hölder's inequality implies

$$\iiint_{\mathcal{V}} \iiint_{Q^n(x,t)} \frac{h(x,t)\, f(\xi,\tau)}{(|x-\xi| + \sqrt{|t-\tau|})^{N+2-\alpha}}$$

$$\leq 2^{n(N+2-\alpha)} \left[\iiint_{\mathcal{V}} \iiint_{Q^n(x,t)} |f(\xi,\tau)|^m \right]^{\frac{1}{p}}$$

$$\cdot \left[\iiint_{\mathcal{V}} \iiint_{Q^n(x,t)} |h(x,t)|^{\frac{p}{p-1}} \right]^{\frac{m-p-q+pq}{pq}}$$

$$\cdot \left[\iiint_{\mathcal{V}} \iiint_{Q^n(x,t)} |f(\xi,\tau)|^q |h(x,t)|^{\frac{p}{p-1}} \right]^{\frac{p-m}{pq}}$$

$$\leq \gamma 2^{n(N+2-\alpha)} \left[(2^n)^{-\frac{N+2}{p}} \|f\|_{L_m(\mathcal{V})}^{\frac{m}{p}} \right]$$

$$\left[(2^n)^{-(N+2)\frac{m-p-q+pq}{pq}} \right] \left[(2^n)^{-\lambda \frac{p-m}{pq}} |f|_{\mathfrak{L}_q^\lambda(\mathcal{V})}^{1-\frac{m}{p}} \right]$$

Thus

$$\begin{aligned}\|Tf\|_{L_p(\mathcal{V})} &\leq \gamma \|f\|_{L_m(\mathcal{V})}^{\frac{m}{p}} |f|_{\mathfrak{L}_q^\lambda(\mathcal{V})}^{1-\frac{m}{p}} \sum_{n=A_\mathcal{V}}^{\infty} 2^{nB} \\ &\leq \gamma \|f\|_{L_m(\mathcal{V})}^{\frac{m}{p}} |f|_{\mathfrak{L}_q^\lambda(\mathcal{V})}^{1-\frac{m}{p}}\end{aligned}$$

where

$$B = (N+2-\lambda)\left[\left(\frac{1}{q} - \frac{\alpha}{N+2-\lambda}\right) - \frac{1}{p}\frac{m}{q}\right] < 0.$$

The second result is proven in roughly the same fashion.